NONSTEADY HEAT-FLUX DISTRIBUTION IN SEMIINFINITE BODIES IN THE PRESENCE OF A FINITE HEAT SOURCE AT THE CONTACT BOUNDARY OF THE BODIES

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On the basis of solving the heat-conduction boundary problem, the nonsteady character of the heat-flux distribution in semiinfinite bodies is analyzed, in the presence of a heat source at the contact boundary of the bodies.

It is known that, in the thermal contact of two semiinfinite rods with different initial temperatures, a constant temperature is established at their contact boundary immediately after contact (disregarding the relaxation time, which is of the order of $10^{-11}$ sec for solids), and does not change over the course of the whole heat-transfer process [1]. It is of interest to elucidate the distribution of the heat fluxes entering each of the semiinfinite bodies when there is a heater at their contact boundary. This investigation is conducted directly for the development and realization of nondestructive methods of determining the thermophysical properties of the material. It is necessary because the realization of absolute methods of determining the thermophysical properties of material on the basis of probing the sample with a known heat flux entails using a symmetric experimental configuration, when a thin heater is placed between two samples with identical thermophysical characteristics. In this case, it is assumed that the heat flux entering the given sample is equal to half the specific power of the heater. It would be more convenient to locate the heater on some base. In this case, a standard material with known thermal properties must be used as the base, or else the heat flux entering the base must be estimated.

Consider the one-dimensional problem of the action of a plane heater (of negligible specific heat) at the point of contact between two semiinfinite rods. The mathematical formulation of the problem is written in the form

$$\frac{\partial T_1(x,\ \tau)}{\partial \tau}=a_1\ \frac{\partial^2 T_1(x,\ \tau)}{\partial x^2},\ \tau>0,\ x>0, \tag{1}$$

$$\frac{\partial T_2(x,\ \tau)}{\partial \tau}=a_2\ \frac{\partial^2 T_2(x,\ \tau)}{\partial x^2},\ \tau>0,\ x<0, \tag{2}$$

$$T_1(x,\ 0)=T_2(x,\ 0)=T_0, \tag{3}$$

$$\left.\begin{aligned} &T_1(0,\ \tau)=T_2(0,\ \tau),\\ &\lambda_2\ \frac{\partial T_2(0,\ \tau)}{\partial x}-\lambda_1\ \frac{\partial T_1(0,\ \tau)}{\partial x}=q(\tau),\end{aligned}\right\} \tag{4}$$

$$T_1(\infty,\ \tau)=T_2(-\infty,\ \tau)=T_0. \tag{5}$$

The solution of the Laplace-transformed equations, with the boundary and initial conditions in Eqs. (3)-(5), takes the form

$$\overline{T}_1(x,\ s)-\frac{T_0}{s}=\overline{q}(s)\ \frac{\exp\left[-\dfrac{x\sqrt{s}}{\sqrt{a_1}}\right]}{(b_1+b_2)\sqrt{s}}, \tag{6}$$

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$$\overline{T}_2(x,\ s)-\frac{T_0}{s}=\overline{q}(s)\ \frac{\exp\left[-\frac{|x|\sqrt{s}}{\sqrt{a_2}}\right]}{(b_1+b_2)\sqrt{s}}, \tag{7}$$

where $b_i = \lambda_i/\sqrt{a_i}$.

From the solution in Eqs. (6) and (7), an expression is written for the heat fluxes in the cross section x = 0

$$q_1(\tau)=q(\tau)\frac{b_1}{b_1+b_2}, \tag{8}$$

$$q_2(\tau)=q(\tau)\frac{b_2}{b_1+b_2}. \tag{9}$$

Thus, immediately after switching on the heater, the heat fluxes entering each of the bodies will reproduce the law specifying the variation over time in the specific power of the heater and will take a value determined by the ratio of the thermal activity of the body and the sum of thermal activities of the two bodies.

It follows from Eqs. (8) and (9) that, with a known thermal activity of one of the semi-infinite bodies and a measured heat flux entering this body, the heat flux from the heater to another body with unknown properties may be determined.

Since the heater always has finite dimensions, it is of interest to elucidate what influence the finiteness of the heater has on the flux distribution. This problem is of equal interest for so-called nondestructive methods of determining the thermophysical characteristics based on two-dimensional solutions of the classical heat-conduction boundary problem with discontinuous boundary conditions of the second kind [2-4].

Consider the following problem. In the contact plane of two semiinfinite bodies bounded by a circle of radius $r_0$, surface heat sources of specific power $w(\tau)$ distributed uniformly over the surface of the circle begin to act at time $\tau = 0$. Beyond the limits of this region, there is no heat transfer between the bodies.

The mathematical formulation of the problem is as follows

$$\frac{\partial T_1}{\partial \tau}=a_1\left(\frac{\partial^2 T_1}{\partial r^2}+\frac{1}{r}\ \frac{\partial T_1}{\partial r}+\frac{\partial^2 T_1}{\partial z^2}\right),\ \tau>0,\ z>0,\ 0\leqslant r<\infty, \tag{10}$$

$$\frac{\partial T_2}{\partial \tau}=a_2\left(\frac{\partial^2 T_2}{\partial r^2}+\frac{1}{r}\ \frac{\partial T_2}{\partial r}+\frac{\partial^2 T_2}{\partial z^2}\right),\ \tau>0,\ z<0,\ 0\leqslant r<\infty, \tag{11}$$

$$T_1(r,\ z,\ 0)=T_2(r,\ z,\ 0)=T_0, \tag{12}$$

$$\begin{cases}\dfrac{\partial T_1(r,\ 0,\ \tau)}{\partial z}=-\dfrac{q_1(r,\ \tau)}{\lambda_1}, & (13)\\[2ex] \dfrac{\partial T_2(r,\ 0,\ \tau)}{\partial z}=\dfrac{q_2(r,\ \tau)}{\lambda_2},\quad 0\leqslant r\leqslant r_0 & (14)\\[2ex] q_1(r,\ \tau)+q_2(r,\ \tau)=w(\tau)\,U(\tau), & (15)\end{cases}$$

$$\frac{\partial T_1(r,\ 0,\ \tau)}{\partial z}=\frac{\partial T_2(r,\ 0,\ \tau)}{\partial z}=0,\ r>r_0, \tag{16}$$

$$T_1(r,\ 0,\ \tau)=T_2(r,\ 0,\ \tau),\ 0\leqslant r<r_0, \tag{17}$$

$$\frac{\partial T_1(0,\ z,\ \tau)}{\partial r}=\frac{\partial T_2(0,\ z,\ \tau)}{\partial r}=0, \tag{18}$$

$$\frac{\partial T_1(\infty, z, \tau)}{\partial r} = \frac{\partial T_2(\infty, z, \tau)}{\partial r} = \frac{\partial T_1(r, \infty, \tau)}{dz} = \frac{\partial T_2(r, -\infty, \tau)}{\partial z} = 0. \tag{19}$$

The solution of Eqs. (10) and (11) with the initial and boundary conditions in Eqs. (12)-(19) for the flux in cross section z = 0 after Laplace transformation takes the form

$$q_1(r, s)\big|_{z=0} = \frac{w(s)\, B(r, s)}{A(r, s) + B(r, s)}, \tag{20}$$

where

$$A(r, s) = \frac{1}{b_1\sqrt{s}} - \frac{2r_0}{\lambda_1\pi}\int_0^\infty \frac{K_1\left(\sqrt{p^2+\frac{s}{a_1}}\,r_0\right) I_0\left(\sqrt{p^2+\frac{s}{a_1}}\,r\right) dp}{\left(\sqrt{p^2+\frac{s}{a_1}}\right)};$$

$$B(r, s) = \frac{1}{b_2\sqrt{s}} - \frac{2r_0}{\lambda_2\pi}\int_0^\infty \frac{K_1\left(\sqrt{p^2+\frac{s}{a_2}}\,r_0\right) I_0\left(\sqrt{p^2+\frac{s}{a_2}}\,r\right) dp}{\left(\sqrt{p^2+\frac{s}{a_2}}\right)}.$$

As $r_0 \to \infty$

$$q_1 = \frac{b_1 w(\tau)}{b_2 + b_1}.$$

Inverse Laplace transformation of Eq. (20) presents considerable difficulties. When $w(\tau) = w_0 = \text{const}$, the inverse transformation from Eq. (20) may be found for the center of the circle (the point z = 0, r = 0). In this case, Eq. (20) is written in the form

$$q_1(s) = K_b \frac{w_0}{s}\, \frac{1 - \exp\left[-r_0\sqrt{\frac{s}{a_2}}\right]}{1 - \exp\left[-r_0\sqrt{\frac{s}{a_1}}\right] + K_b\left(1 - \exp\left[-r_0\sqrt{\frac{s}{a_2}}\right]\right)}, \tag{21}$$

where $K_b = b_1/b_2$.

The inverse Laplace transformation for the point z = 0, r = 0 is written in terms of dimensionless numbers

$$Q_1 = \frac{q_1(\mathrm{Fo}_1)}{w_0} = \frac{K_b}{1+K_b}\sum_{n=0}^{\infty}\sum_{j=0}^{n}\frac{n!}{j!\,(n-j)!}\,\frac{K_b^j}{(1+K_b)^n}\times$$
$$\times\left[\operatorname{erfc}\frac{n + j(\sqrt{K_a}-1)}{2\sqrt{\mathrm{Fo}_1}} - \operatorname{erfc}\frac{n+\sqrt{K_a}+j(\sqrt{K_a}-1)}{2\sqrt{\mathrm{Fo}_1}}\right], \tag{22}$$

where $K_a = a_1/a_2$; $\mathrm{Fo}_1 = a_1\tau/r_0^2$.

As $\mathrm{Fo}_1 \to \infty$

$$\lim_{\mathrm{Fo}_1\to\infty}\frac{q_1(\mathrm{Fo}_1)}{w_0} = \frac{K_b\sqrt{K_a}}{1+K_b\sqrt{K_a}} = \frac{\lambda_1}{\lambda_1+\lambda_2}.$$

As $\mathrm{Fo}_1 \to \infty$

$$\lim_{\mathrm{Fo}_1\to 0}\frac{q_1(\mathrm{Fo}_1)}{w_0} = \frac{K_b}{1+K_b}.$$

Calculations for various ratios of $K_a$ and $K_b$ have been performed using Eq. (22) on the EC-1020 computer (Fig. 1). The values of the thermophysical characteristics are taken from [5, 6].

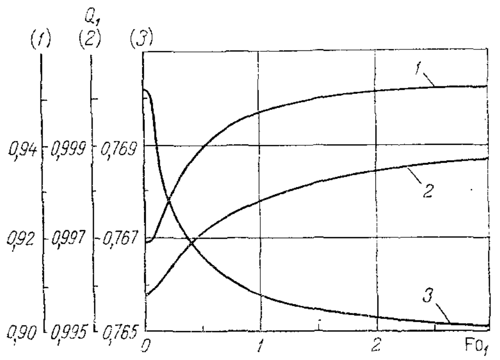


Fig. 1

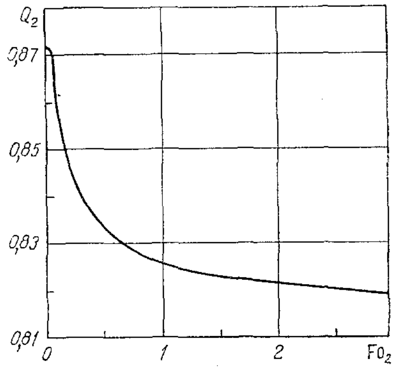


Fig. 2

Fig. 1. Dependence of the dimensionless flux $Q_1$ on the Fourier number $Fo_1$:1) $K_b = 11.41$, $K_a = 4.203$ (glass–rubber); 2) 237.4, 1246 (copper–rubber); 3) 3.351, 0.9348 (polymethylmethacrylate–rubber). For all three curves the origin of the abscissa is at point 0.

Fig. 2. Dependence of the flux ratio $Q_2$ on the Fourier number $Fo_2$ for $K_b = 1.147$, $K_a = 1.16$ (polystyrene PPS–polymethylmethacrylate).

It is evident from the given results that the maximum redistribution of the fluxes over time is observed on the initial section (up to $Fo_1 = 1$), i.e., precisely in the region which is often used in nonsteady methods of determining the thermophysical characteristics. Note also that, when $K_a = 1$, $K_b = 1$, it follows directly from Eq. (22) that $Q_1 = 0.5$ (the thermophysical properties are equal); when $K_a = 1$, $K_b \neq 1$, $Q_1 = K_b/(1 + K_b)$, i.e., the heat flux does not depend on the time. This is natural, since it is the thermal diffusivity which determines the character of transient thermal processes.

In [3, 7, 8] it was supposed that heat fluxes entering the semiinfinite bodies from a finite heater of constant power at their contact plane are constant. As shown by the present results, this assumption may lead to a large error in determining the thermophysical characteristics. For example, in [8], the error in determining the thermal conductivity because of this factor may be more than 5%.

To establish the validity of the theoretical relations in the method of determining the thermophysical characteristics of materials, a physicomathematical model which assumes that the ratio of the fluxes $q_1$ and $q_2$ entering semiinfinite regions of the sample and standard bodies depeds on the thermal activities of the bodies and not on the time was considered in [9].

The analytical expression for calculating the ratio $q_2(Fo_2)/q_1(Fo_2)$ at the center of the circle ($z = 0$, $r = 0$) is written in the form

$$Q_2 = \frac{q_2(\mathrm{Fo}_2)}{q_1(\mathrm{Fo}_2)} = \left\{ \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{n!}{j!\,(n-j)!} \frac{K_b^{j+1}}{(1+K_b)^{n+1}} \times \right.$$
$$\left. \times \left[ \operatorname{erfc} \frac{n + j(\sqrt{K_a} - 1)}{2\sqrt{K_a \mathrm{Fo}_2}} - \operatorname{erfc} \frac{n + j(\sqrt{K_a} - 1) + \sqrt{K_a}}{2\sqrt{K_a \mathrm{Fo}_2}} \right] \right\}^{-1} - 1, \qquad (23)$$

where $Fo_2 = a_2\tau/r_0^2$.

The curve of $q_2(Fo_2)/q_1(Fo_2)$ shown in Fig. 2 for the case of contact when semiinfinite bodies differing in their thermophysical characteristics are in contact with a circular heat source clearly shows that this dependence is unsteady in character. Only in the case when $K_a = 1$ is the ratio of fluxes $q_1$ and $q_2$ independent of the time; see Eq. (23). In the general case, when $K_a \neq 1$, the ratio $q_2(Fo_2)/q_1(Fo_2)$ depends weakly on the time only in the initial (one-dimensional) stage of development of a temperature field at the center of a circular source ($Fo_2 \leq 0.05$). Subsequently, this dependence has a more clearly expressed nonsteady character.

Note that

$$\lim_{Fo_2\to\infty}\frac{q_2(Fo_2)}{q_1(Fo_2)}=\frac{\lambda_2}{\lambda_1},\quad \lim_{Fo_2\to 0}\frac{q_2(Fo_2)}{q_1(Fo_2)}=\frac{b_2}{b_1}.$$

In [3], in order to preserve the good agreement of the experimental data and the theoretical curves, it was required that the properties of the standard body be as close as possible to those of the sample being investigated, without explaining why this requirement arises. It is obvious that the increase in measurement error when the thermophysical properties of the standard differ from those of the sample is associated with violation of the boundary conditions ($q_1 \neq$ const, $q_2 \neq$ const). Note also that, even approximate equality of the properties of the standard and the sample is very difficult to ensure in practice, since a very limited number of standard materials exists at present [10].

## NOTATION

x, r, z, coordinates; $\tau$, time; $r_0$, radius of heating spot; $q(\tau)$, specific heat flux; $T_0$, initial temperature; $w_0$, $w(\tau)$, constant and variable specific power of surface heat sources in the heating spot; $U(\tau)$, Heaviside unit step function; $a_1$, $a_2$, $\lambda_1$, $\lambda_2$, $b_1$, $b_2$, thermal diffusivity, thermal conductivity, and thermal activity of the first and second semiinfinite bodies, respectively; $I_0(x)$, $K_1(x)$, zero-order modified Bessel function of the first kind and first-order Bessel function of the second kind, respectively; erfc(x), additional probability integral; $T_1(x, \tau)$, $T_2(x, \tau)$, $q_1(\tau)$, $q_2(\tau)$, temperatures of the first and second bodies and heat fluxes entering the first and second bodies for a one-dimensional problem; $T_1(r, z, \tau)$, $T_2(r, z, \tau)$, $q_1(Fo)$, $q_2(Fo)$, temperatures of the first and second bodies and the heat fluxes at the center of the circle entering the first and second bodies for a two-dimensional problem; $Q_1 = q_1(Fo_1)/w_0$, dimensionless flux; $Q_2 = q_2(Fo_2)/q_1(Fo_2)$, flux ratio.

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